THE HYPOTHESIS OF LOCALNESS IN THE TURBULENT MOTION OF A VISCOUS FLUID

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PMM Vol.22, No.5, 1958, pp. 600-611

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(Received 2 June 1958)

1. The present status of the question. In the semi-empirical theories which are widely applied at the present time for turbulent motion and heat and mass transfer, the total fluid flow is divided schematically into sharply distinguished regions where friction and heat and mass transfer have either a purely molecular character (laminar sublayer) or a purely molar character (turbulent core). An intermediate transition region (sometimes called the "buffer" region) may be introduced to increase the accuracy of the theory of heat and mass transfer, but the laws governing this region have been studied only superficially and have usually been replaced by approximate empirical relationships.

Experiments defining the mean velocity near a body surface, among them some pertaining to recent times [1], are presented in Fig. 1. From these data it is seen that there is a transition (*b*) from the linear velocity profile immediately at the surface, represented by the curved line (*a*) in this semi-logarithmic plot, to a logarithmic profile far from the surface of the body - the straight line (*c*). All three regions have been included in analytic expressions for the velocity profile obtained in various theoretical investigations. We will begin with a reference to the earliest work in this direction, by the Japanese scientist, Wada [2] in 1927. For the calculation of the effect of viscosity on the mechanism of turbulent friction he proposed the formula

$$\tau = \mu \frac{du}{dy} + \rho l^2 \left(\frac{du}{dy}\right)^2 \qquad (1.1)$$

which expresses a simple superposition of purely laminar (molecular) friction and turbulent (molar) friction, the latter being independent of the molecular viscosity. The first term in this formula represents the law of Newton, and the second the formula of Prandtl. Both terms are appropriate for the simplest steady rectilinear motion parallel to a plane.

The validity of formula (1.1) is debatable, inasmuch as the result of molar transport of momentum, described by the second term in the well-known formula of Reynolds

$$\tau = \mu \frac{du}{dy} + (-\rho \overline{u'v'}) \tag{1.2}$$

contains, in addition, an inherent influence of molecular viscosity which is essential for the transition region but is not taken into account in Prandtl's formula. Thus, the mutual influence of the processes of molecular and molar momentum transport is reduced in the relationship (1.1) to a simple superposition.

The formula (1.1) has formed the starting point for all subsequent investigations in this direction, and in particular for contributions by Szablewski [3], van Driest [4] and Miles [5] which have recently appeared in foreign publications. The shortcomings of formula (1.1) are concealed in these papers because of unavoidable adjustments made by the authors in the law for the variation of the "mixing length" l. Thus, for example, the second of the above authors employs, instead of the simple and well-known formula of Prandtl, $l = \kappa y$, a considerably more complicated law

$$l = \varkappa y\,(1 - e^{y/A})$$

which contains a new empirical constant A and whose use is excused by artificial considerations concerning increased damping of the fluctuation close to the solid surface. The other authors, with the same object, also distort the initial development of the quantity l in a certain experimentally determined interval near the wall.

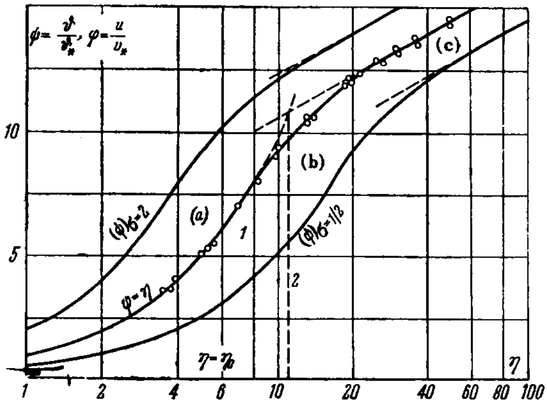

Fig. 1.

The course of the present investigation is in principle different. What is proposed is an extension, to the whole field of turbulent motion where there is an interaction of molecular and molar processes, of the hypothesis of localness for the mechanism of turbulent mixing and for the Reynolds analogy between the transport of momentum and heat. This allows the establishment of a single point of view for the whole existing semi-empirical theory and the derivation of new formulas for friction and heat transfer in turbulent motion Application of these formulas determines the velocity profile and the excess temperature in terms of analytic expressions which are continuous and have continuous first derivatives normal to the direction of flow throughout the laminar, transition, and completely turbulent regions.

2. The hypothesis of localness in contemporary semi-empirical theories of turbulence. A characteristic feature of the semi-empirical theories of turbulence commonly accepted at the present time is an assumption concerning their differential nature, in that the mechanism of purely turbulent (molar) momentum transport is assumed to be completely determined when local values are assigned for the physical constants of the fluid and for the derivatives of the mean velocity along the coordinate normal to the direction of flow. The absolute magnitude of the mean velocity at a given point in a steady flow has no effect on the turbulent transport mechanism, being equivalent to the velocity of a uniformly moving system of coordinates which may be mentally associated with the moving layer under consideration. Moreover, it is assumed that at a sufficient distance from the solid surface the molar exchange dominates over the molecular exchange to such an extent that there is no great error in neglecting the ordinary viscosity and heat conductivity by comparison with their turbulent analogs.

These assumptions in the aggregate make up the content of a hypothesis underlying all modern semi-empirical theories of turbulence, a hypothesis which might be termed "the hypothesis of localness for the turbulent transport mechanism".

In contrast to the differential approach just described, none of the alternative "integral" formulations of turbulent transport, which take into account the influence of processes occurring far away from a given point in a turbulent flow, have so far been usefully expressed in concrete form.

It is well known [6] that the semi-empirical formulas of Prandtl and Karman may be immediately derived on the basis of the specified hypothesis of localness together with simple dimensional arguments. If it is assumed as a "first approximation"* that the local variation of mean velocity is defined by a single first derivative du/dy, then dimensional considerations necessarily lead to the introduction of a certain length $l(y)$ - the

* The notion of first, second, etc. approximations is arbitrary here.

Prandtl "mixing length" - as an additional concept without which it is not possible within this approximation to construct a formula for the shearing stress according to the hypothesis of localness. On introducing the length in question, one may at once convince oneself by simple dimensional considerations that under these conditions there exists only one possible combination expressing the shearing stress τ in terms of the density of the fluid ρ, the "mixing length" l, and the derivative du/dy;

$$\tau = \rho l^2 \left(\frac{du}{dy}\right)^2 \tag{2.1}$$

At the same time, of course, a quantitative expression for the dependence $l(y)$ follows out of certain further arguments which are not part of the hypothesis of localness in the "first approximation".

Using the "second approximation", i.e. assigning changes in mean velocity to the first two derivatives $u'(y)$ and $u''(y)$, we infer from the same dimensional considerations the existence and uniqueness of Karman's formula for the shearing stress,

$$\tau = \rho \varkappa^2 \frac{u'^4}{u''^2} \tag{2.2}$$

where κ is a certain dimensionless constant.

Comparing formulas (2.1) and (2.2), which refer to different "approximations"–and we will stress this point–the well-known formula of Karman is obtained,

$$l = -\varkappa \frac{u'}{u''} \tag{2.3}$$

However, it seems to us more correct to think that the theory of the "second approximation" does not require the introduction of a "mixing length" l, which is a quantity foreign to the phenomenological theories under consideration.

Application of the semi-empirical theories to processes of turbulent heat transfer is based on the so-called "Reynolds analogy", which is based in turn on the community of momentum and heat with their carrier. According to this analogy it may be assumed that the dynamic coefficient of turbulent mixing A and the kinematic coefficient $\epsilon = A/\rho$ for the finite volumes of fluid participating in the turbulent mixing have the same value in transport processes for momentum and for heat. Such an assumption presupposes the absence of effects caused by changes in the heat content of the flow in the turbulent mixing mechanism (the hypothesis of inertness, as regards the turbulent structure of the flow, for a quantity transferred with the fluid) and probably is correct for not too large changes in temperature.

The analogy of Reynolds may be represented quantitatively as

$$\tau = \rho\varepsilon \frac{du}{dy}, \qquad q = \rho c_p \varepsilon \frac{d\vartheta}{dy} \qquad \text{or} \qquad \frac{q}{\tau} = c_p \frac{d\vartheta}{du}. \tag{2.4}$$

Denoting the coefficient of molecular heat conductivity by λ, we have the equalities

$$\tau = \mu \frac{du}{dy}, \qquad q = \lambda \frac{d\vartheta}{dy} \quad \text{or} \quad \frac{q}{\tau} = \frac{c_p}{\sigma} \frac{d\vartheta}{du} \qquad \left(\sigma = \frac{\mu c_p}{\lambda}\right) \tag{2.5}$$

A comparison of formulas (2.4) and (2.5), in which the "Reynolds analogy" is expressed for turbulent and for laminar motion, shows that the ratio q/τ will be identical in the two cases if the Prandtl number σ is equal to unity.

3. The hypothesis of localness and the Reynolds analogy when there is an interaction between molecular and molar exchange. We will now enlarge the domain of application of the hypothesis of localness by giving up the assumption which mas made in the formulation of Section 2 concerning the absence of molecular effects in molar transport. In other words, the influence of the viscosity and heat conductivity of the fluid on the turbulent transport mechanism will be considered. According to the hypothesis of localness, this influence should be expressed by the introduction into the friction and heat transfer laws of additional factors incorporating functions of the local Reynolds or Peclet number. According to this same hypothesis of localness, we will understand by the local Reynolds number a dimensionless combination of the physical constants of the fluid, density and viscosity, together with quantities specifying the variation of mean velocity. This combination will be inversely proportional to the first power of the viscosity coefficient. It is easy to contrive a relationship to serve as the required combination,

$$R = \frac{\varepsilon}{\nu} \tag{3.1}$$

in which the quantity ϵ, having the dimensions of kinematic viscosity and representing a combination of the quantities l and du/dy in the "first approximation", of $u'(y)$ and $u''(y)$ in the "second approximation", and so on, is seen from the form of its dependence on these quantities to be nothing else but the kinematic turbulent exchange coefficient for the case of zero viscosity. Thus we will have in the "first approximation" and in the "second approximation" respectively*

$$R = \frac{l^2}{\nu} \frac{du}{dy}, \qquad R = \frac{\varkappa^2 u'^3}{\nu u''^2} \tag{3.2}$$

* The use of the concept of local Reynolds number in the "first approximation" apparently originated in our papers on the application of the theory of similitude to turbulent flow [7].

As usual, we will understand by the local Peclet number P the product of local Reynolds number and Prandtl number; that is, we will put

$$P = \sigma R = \frac{\rho c_p \varepsilon}{\lambda} \tag{3.3}$$

Then the following formulas for friction and heat transfer appear as a quantitative expression of the generalized hypothesis of localness:

$$\tau = \mu \frac{du}{dy} f(R) = \rho\nu \frac{du}{dy} f\left(\frac{\varepsilon}{\nu}\right) \tag{3.4}$$

$$q = \lambda \frac{d\vartheta}{dy} f_1(P) = \lambda \frac{d\vartheta}{dy} f_1(\sigma R) = \lambda \frac{d\vartheta}{dy} f_1\left(\frac{\rho c_p \varepsilon}{\lambda}\right) \tag{3.5}$$

At this point all that can be said concerning the functions f and f_1 is that each is equal to unity in the region where the motion and heat transfer are laminar, in accordance with the laws of Newton and Fourier (2.5). Each is also equal to its argument in the region of fully turbulent exchange, for only then do the quantities ν and λ of molecular origin drop out of formulas (3.4) and (3.5) so that these formulas take on the form (2.4).

It follows that the functions f and f_1 coincide when their arguments vary in the regions $0 \leqslant R \leqslant R_0$ and $0 \leqslant \sigma R \leqslant \sigma R_0$, where R_0 represents a critical local Reynolds number corresponding to the boundary of the region of laminar motion, and that each function tends asymptotically to its argument for indefinitely large values of these arguments.

It is natural to suppose that the function $f(R)$, representing the ratio of total friction to laminar friction in the transition region, ought to increase sharply beginning at the point $R = R_0$ where molar transfer first emerges as much more important than molecular transfer. Such behavior may be imparted to the function $f(R)$ by using the segment AL (Fig. 2) of one branch of a rectangular hyperbola with asymptotes $f = \pm R$. Then an analytic expression for the function $f(R)$ over the whole interval of change in R will be provided by

$$f(R) = \begin{cases} 1 & \text{for } 0 \leqslant R \leqslant R_0 \\ \sqrt{R^2 - R_0^2 + 1} & \text{for } R_0 \leqslant R \end{cases} \tag{3.6}$$

Repeating exactly the same reasoning, we will take for the function $f_1(\sigma R)$ the analytic expression

$$f_1(\sigma R) = \begin{cases} 1 & \text{for } 0 \leqslant \sigma R \leqslant \sigma R_0 \\ \sqrt{\sigma^2 R^2 - \sigma^2 R_0^2 + 1} & \text{for } \sigma R_0 \leqslant \sigma R \end{cases} \tag{3.7}$$

This shows that if the turbulent mixing coefficients ϵ (or A) for transport of momentum and heat are considered as identical in the treatment adopted here for the Reynolds analogy in the region of purely molar transfer, then the functions f and f_1 may also be considered as identical.

Returning to formulas (3.4) and (3.5), we obtain the following extended formulation of the Reynolds analogy;

$$\frac{q}{\tau}=\frac{\lambda}{\mu}\frac{f(\sigma R)}{f(R)}\frac{d\vartheta}{du}=\frac{c_p}{\sigma}\frac{f(\sigma R)}{f(R)}\frac{d\vartheta}{du} \tag{3.8}$$

It is immediately seen that formula (2.5) will be correct in regions where processes for transfer of momentum and heat have a purely molecular character ($f = 1$), and on the other hand, that formula (2.4) will be correct in regions of purely molar transfer [$f(R) = R$, $f(\sigma R) = \sigma R$]. The rule just obtained (3.8) shows how the Prandtl number σ enters into the ratio of heat transfer to friction in transition regions where molecular and molar processes interact.

Turning once more to formular (3.4) and (3.5) and substituting for the functions f and f_1 their values from (3.6) and (3.7), we obtain the following final expressions for τ and q;

$$\tau=\begin{cases}\mu\, du/dy & \text{for } 0\leqslant R\leqslant R_0\\ \mu\,(du/dy)\sqrt{R^2-R_0^2+1} & \text{for } R_0\leqslant R\end{cases} \tag{3.9}$$

$$q=\begin{cases}\lambda\,(d\vartheta/dy) & \text{for } 0\leqslant \sigma R\leqslant \sigma R_0\\ \lambda\,(d\vartheta/dy)\sqrt{\sigma^2R^2-\sigma^2R_0^2+1} & \text{for } \sigma R_0\leqslant \sigma R\end{cases} \tag{3.10}$$

Here $R = \epsilon/\nu$, but the expression for ϵ depends on the approximation chosen through the local derivatives of the mean velocity (and the length l).

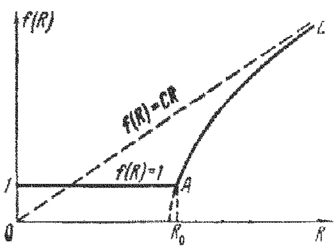

Fig. 2.

4. Determination of the velocity profile for uniform motion according to the "first approximation". Stopping at the "first approximation" and considering for simplicity the case of steady turbulent motion near a smooth plane in the absence of longitudinal pressure gradients, we will put, following Prandtl [8]:

$$\tau=\text{const}=\tau_w, \qquad l=\varkappa y \tag{4.1}$$

where τ_w is the shearing stress on the smooth surface and κ is a constant to be determined experimentally. Then we will have

$$\varepsilon = l^2 \frac{du}{dy} = \varkappa^2 y^2 \frac{du}{dy}$$

and at the edge of the laminar region, where $y = y_0$,

$$\varepsilon_0 = \varkappa^2 y_0^2 \left(\frac{du}{dy}\right)_0 \tag{4.2}$$

Equation (3.9) is then broken up into two parts,

$$\tau_w = \mu \frac{du}{dy} \qquad \text{for } 0 \leqslant y \leqslant y_0 \tag{4.3}$$

$$\tau_w = \mu \frac{du}{dy}\left[\frac{\varkappa^4 y^4}{\nu^2}\left(\frac{du}{dy}\right)^2 - \frac{\varkappa^4 y_0^4}{\nu^2}\left(\frac{du}{dy}\right)_0^2 + 1\right]^{1/2} \qquad \text{for } y_0 \leqslant y \tag{4.4}$$

We now introduce universal coordinates

$$\varphi = \frac{u}{v_*}, \qquad \eta = \frac{y v_*}{\nu}, \qquad v_* = \sqrt{\frac{\tau_w}{\rho}} \tag{4.5}$$

in terms of which the equations assume the form

$$\frac{d\varphi}{d\eta} = 1 \qquad \text{for } 0 \leqslant \eta \leqslant \eta_0 \tag{4.6}$$

$$\frac{d\varphi}{d\eta}\left[\varkappa^4 \eta^4 \left(\frac{d\varphi}{d\eta}\right)^2 - \varkappa^4 \eta_0^4 \left(\frac{d\varphi}{d\eta}\right)_0^2 + 1\right]^{1/2} = 1 \qquad \text{for } \eta_0 \leqslant \eta \tag{4.7}$$

From the first it follows that

$$\varphi = \eta \qquad \text{for } 0 \leqslant \eta \leqslant \eta_0 \tag{4.8}$$

Noting that $(d\phi/d\eta)_0 = 1$, and solving the second equation for $d\phi/d\eta$, we obtain

$$\frac{d\varphi}{d\eta} = \left[\frac{\varkappa^4\eta_0^4 - 1 \pm \sqrt{(\varkappa^4\eta_0^4 - 1)^2 + 4\varkappa^4\eta^4}}{2\varkappa^4\eta^4}\right]^{1/2}$$

or, retaining naturally only the upper sign and somewhat rearranging the right-hand side,

$$\frac{d\varphi}{d\eta} = \left[\frac{\varkappa^4\eta_0^4 - 1}{2\varkappa^2\eta^2} + \left(\frac{(\varkappa^4\eta_0^4 - 1)^2}{4\varkappa^4\eta^4} + 1\right)^{1/2}\right]^{1/2} \frac{1}{\varkappa\eta} \tag{4.9}$$

Making the substitution

$$t = \left[\frac{\varkappa^4\eta_0^4 - 1}{2\varkappa^2\eta^2} + \left(\frac{(\varkappa^4\eta_0^4 - 1)^2}{4\varkappa^4\eta^4} + 1\right)^{1/2}\right]^{1/2} \qquad (t_0 \geqslant t \geqslant 1)$$
$$t = t_0 = \varkappa\eta_0 \quad \text{for } \eta = \eta_0, \qquad t = 1 \quad \text{for } \eta = \infty \tag{4.10}$$

we arrive at a quadrature

$$\varphi = \eta_0 + \frac{1}{\varkappa}(t_0 - t) + \frac{2}{\varkappa}\int_t^{t_0} \frac{dt}{t^4 - 1} \tag{4.11}$$

in which the boundary condition of the problem has already been satisfied:

$$\varphi = \eta_0 \quad \text{for } \eta = \eta_0 \tag{4.12}$$

Performing the integration on the right-hand side of (4.11), we find the following velocity distribution in universal coordinates;

$$\varphi = \begin{cases} \eta & \text{for } 0 \leqslant \eta \leqslant \eta_0 \\ \eta_0 + \varkappa^{-1}[\theta(t_0) - \theta(t)] & \text{for } \eta \leqslant \eta_0 \end{cases} \tag{4.13}$$

where

$$\theta(t) = t - \frac{1}{2}\ln\frac{t+1}{t-1} - \tan^{-1} t \tag{4.14}$$

This velocity distribution is immediately seen from the formula to be continuous at the point $\eta = \eta_0$. Evaluating the right-hand side of (4.9) at $\eta = \eta_0$, we may satisfy ourselves that at this point $d\phi/d\eta = 1$, so that the proposed distribution is not only continuous at the point $\eta = \eta_0$ but also has a continuous first derivative at this point.

It remains to determine the constants κ and η_0. The fact deserves attention that in the "first approximation" a study of the transition region does not increase the number of empirical constants.

In order to determine κ and η_0 we will work out the asymptotic expression for $\phi(\eta)$ corresponding to $\eta \to \infty$ or $t \to 1$. We will have, according to (4.10) and (4.14), the following asymptotic equality;

$$t - 1 \sim \frac{\varkappa^4\eta_0{}^4 - 1}{4\varkappa^2\eta^2}, \quad \theta(t) \sim 1 - \tfrac{1}{2}\ln 2 - \tfrac{1}{4}\pi + \tfrac{1}{2}\ln(t-1) \tag{4.15}$$

whereupon it is not difficult to find an asymptotic expression for the function $\phi(\eta)$,

$$\varphi(\eta) \sim \frac{1}{\varkappa}\ln\eta + C(\eta_0, \varkappa) \tag{4.16}$$

with

$$C(\eta_0, \varkappa) = \eta_0 + \frac{1}{\varkappa}(t_0 - 1) - \frac{1}{2\varkappa}\ln\frac{t_0+1}{t_0-1} - \frac{1}{2\varkappa}\ln\left(\frac{\varkappa^2\eta_0{}^4}{8}\right) - \\ - \frac{1}{\varkappa}\tan^{-1} t_0 + \frac{\pi}{4\varkappa} \tag{4.17}$$

The asymptotic equality (4.16) is nothing else but the well-known logarithmic velocity law. Putting this law in the form

$$\varphi = 5.6\log\eta + 4.9 \tag{4.18}$$

which is apparently more accurate [1] than the formula of Nikuradse, and comparing (4.16) with (4.18), we obtain on rounding off the value of η_0

$$\varkappa = 0.41, \qquad \eta_0 = 7 \tag{4.19}$$

It is self-evident that these constants could have been determined by a fit to the logarithmic formula of Nikuradse,

$$\varphi = 5.75 \log \eta + 5.5$$

On specifying the constants, we obtain the following analytic expression for the velocity profile in the whole field:

$$\varphi = \begin{cases} \eta \quad \text{for } 0 \leqslant \eta \leqslant 7 \\ 10.14 - 2.44\left(t - \frac{1}{2}\ln\frac{t+1}{t-1} - \tan^{-1} t\right) \\ t = \left\{\frac{202}{\eta^2} + \left[\left(\frac{202}{\eta^2}\right)^2 + 1\right]^{1/2}\right\}^{1/2} \quad \text{for } \eta \geqslant 7 \end{cases} \tag{4.20}$$

A comparison of results calculated according to formula (4.20) with experimental data [1] (Fig. 1)* leads to a completely satisfactory agreement.

The computation of the velocity profile becomes a rather more complicated matter, if one takes into account the known linear variation of shearing stress normal to the flow in a pipe, or uses a more complicated law for the variation of the quantity l, or considers a parameter containing some measure of longitudinal pressure gradient. The existence of good agreement between the experimental data and the simple velocity profile (4.20), which does not consider these effects, is explained by the relatively small extent (about 20 per cent of the pipe radius) of the region where the velocity profile changes from linear to logarithmic, and by the weakness of the subsequent deviation of the profile from the logarithmic law.

We will pause now to consider cases for which the distribution of friction across the flow is not known beforehand (boundary layer, jet, wake, etc.), so that it is necessary to use the general equations of mean turbulent motion in the form given by Reynolds, equations which contain the derivative $\partial\tau/\partial y$ of the shearing stress along a coordinate normal to the flow. The application of formulas (3.9) and (3.10) in these cases is not admissible on the grounds, firstly, that differentiation of the approximate formulas may lead to significant errors, and secondly, that the appearance of higher velocity derivatives is scarcely permissible in expressions where, depending on the number of the approximation, derivatives of corresponding order have previously been discarded. It is more correct in these cases to apply the hypothesis of localness to the derivative $\partial\tau/\partial y$ directly, putting

* Although the experimental points in Fig. 1 are not identified, they represent, as a matter of fact, the results of measurements by different authors (Laufer, Schultz-Grunow and others). We have taken this set of points from Fig. 4 of the survey by Clauser [1].

$$\frac{\partial\tau}{\partial y} = \mu\,\frac{\partial^2 u}{\partial y^2} f(R) \tag{4.21}$$

where the function f and the local Reynolds number R have the same meaning as before. Thus in the case of the "first approximation" we will have

$$\frac{\partial\tau}{\partial y} = \mu\,\frac{\partial^2 u}{\partial y^2}\, f\,\frac{(l^2\ \partial u/\partial y)}{\nu} \tag{4.22}$$

It is easily seen that an asymptotic expression for this formula corresponding to the assumption about the absence of effects of molecular viscosity is provided by the well-known formula of Taylor,

$$\frac{\partial\tau}{\partial y} = \rho l^2\,\frac{\partial u}{\partial y}\,\frac{\partial^2 u}{\partial y^2} \tag{4.23}$$

which is known to give results sometimes agreeing with and sometimes differing from those of Prandtl's theory.

According to the considerations stated in the present article the formula of Taylor (4.23), together with its generalizations contained in the general formula (4.21), should occupy an independent position in the semi-empirical theories of turbulence. This question will constitute the subject of a separate investigation.

5. Determination of the uniform temperature profile according to the "first approximation". For the case of velocity and temperature independent of the longitudinal coordinate, if follows from the equations of mean motion and mean heat transfer that one may take

$$\tau = \tau_w, \qquad q = q_w \tag{5.1}$$

where q_w is the rate of heat transfer per unit time per unit surface area for a body immersed in the fluid.

In addition to the universal scales for velocity v_* and length $l_* = \nu/v_*$ considered earlier, we will introduce a scale for temperature θ_* together with a dimensionless quantity ψ,

$$\psi = \frac{\vartheta}{\vartheta_*} \qquad \left(\vartheta_* = \frac{q_w}{\rho c_p v_*}\right) \tag{5.2}$$

Putting, as before,

$$l = \varkappa y, \qquad R = \frac{l^2}{\nu}\,\frac{du}{dy} = \frac{\varkappa^2 y^2}{\nu}\,\frac{du}{dy} = \varkappa^2\eta^2\,\frac{d\varphi}{d\eta}. \tag{5.3}$$

we may rewrite the basic equations (3.9) and (3.10) in the form

$$1 = \begin{cases} \dfrac{d\varphi}{d\eta} & \text{for } 0 \leqslant \eta \leqslant \eta_0 \\ \dfrac{d\varphi}{d\eta}\left[\varkappa^4\eta^4\left(\dfrac{d\varphi}{d\eta}\right)^2 - \varkappa^4\eta_0^4\left(\dfrac{d\varphi}{d\eta}\right)_0^2 + 1\right]^{1/2} & \text{for } \eta_0 \leqslant \eta \end{cases} \tag{5.4}$$

$$1 = \begin{cases} \dfrac{1}{\sigma}\dfrac{d\psi}{d\eta} & \text{for } 0 \leqslant \eta \leqslant \eta_0 \\ \dfrac{1}{\sigma}\dfrac{d\psi}{d\eta}\left[\sigma^2\varkappa^4\eta^4\left(\dfrac{d\varphi}{d\eta}\right)^2 - \sigma^2\varkappa^4\eta_0^4\left(\dfrac{d\varphi}{d\eta}\right)_0^2 + 1\right]^{1/2} & \text{for } \eta_0 \leqslant \eta \end{cases} \tag{5.5}$$

It is readily seen that if a new independent variable $\zeta = \sigma\eta$ is introduced in the system of equations (5.5) they take on the form

$$1 = \frac{d\psi}{d\zeta} \qquad \text{for } 0 \leqslant \zeta \leqslant \zeta_0$$
$$1 = \frac{d\psi}{d\zeta}\left[\varkappa^4\zeta^4\left(\frac{d\varphi}{d\zeta}\right)^2 - \varkappa^4\zeta_0^4\left(\frac{d\varphi}{d\zeta}\right)_0^2 + 1\right]^{1/2} \text{for } \zeta_0 \leqslant \zeta \tag{5.6}$$

Comparing the systems (5.6) and (5.4), the boundary conditions

$$\varphi = 0 \quad \text{for } \eta = 0, \qquad \psi = 0 \quad \text{for } \zeta = 0 \tag{5.7}$$

in the laminar region of flow, and the conditions for joining the solutions at the edge of this region,

$$\varphi = \eta_0 \quad \text{for } \eta = \eta_0, \qquad \psi = \zeta_0 \quad \text{for } \zeta = \zeta_0 \tag{5.8}$$

we conclude that the following equation will be satisfied in the entire field of flow.

$$\psi = \frac{\vartheta}{\vartheta_*} = \varphi(\sigma\eta) \tag{5.9}$$

Thus, the universal distribution of dimensionless temperature for a given value of the Prandtl number may easily be constructed from the known velocity distribution (4.20). The method of construction is based on the property established by the relationship (5.9); the value of the ordinate for the temperature ratio $\psi = \theta/\theta_*$ at a point with abscissa η is equal to the value of the ordinate for the dimensionless velocity $\phi = u/v_*$ at a point with abscissa $\sigma\eta$.

Thus (Fig. 1) the curve for the dimensionless temperature distribution θ/θ_* lies above and to the left of the curve for the dimensionless velocity u/v_* if $\sigma > 1$ and below and to the right of this same curve if $\sigma < 1$. For $\sigma = 1$ the curves coincide. As an example, the position of the temperature curves is shown in Fig. 1 for the values $\sigma = 1/2$, 1, 2. Curves of similar type but with breaks at the boundaries of the transition region are obtained in the well-known theory of Karman [9].

6. Remarks on the velocity profile according to the "second approximation". As has already been shown in Section 3, the hypothesis of localness in the "second approximation" is expressed by the formula

$$\tau = \mu u' f(R) \qquad \left(R = \frac{\varkappa^2 u'^3}{\nu u''^2}\right) \tag{6.1}$$

where R is the local Reynolds number.

In the laminar region the function f remains the same as in the case of the "first approximation"; i.e. equal to unity, but in the turbulent region it tends with increasing local Reynolds number to the linear function

$$f\left(\frac{\varkappa^2 u'^3}{\nu u''^2}\right) \rightarrow \varkappa^2 \frac{u'^3}{\nu u''^2} \tag{6.2}$$

where κ^2 is an unknown coefficient of proportionality which has to be determined experimentally. On substituting this value of the function in the basic relationship (6.1) we obtain Karman's formula (2.2), which, like the earlier formula of Prandtl (2.1), may be considered as an asymptotic expression for the general stress formula

$$\tau = \begin{cases} \mu u' & \text{for } 0 \leqslant R \leqslant R_0 \\ \mu u' \sqrt{R^2 - R_0^2 + 1} & \text{for larger } R \end{cases} \qquad \text{for } R_0 \leqslant R] \tag{6.3}$$

In this case the question of joining the solution in the laminar region with the remaining portion of the flow presents a certain difficulty. The local definition (6.1) for the Reynolds number R assumes *a priori* that $u'' \neq 0$, but this condition is not fulfilled in the laminar region. One may proceed in one of the following two ways: either, by abandoning the continuous variation of the Reynolds number and inserting for the turbulent region the initial Reynolds number (here and in the sequel primes denote derivatives of ϕ with respect to the universal coordinate η, whereupon $\phi_0' = 1$)

$$R_0 = \frac{\varkappa^2 u_0'^3}{\nu u_0''^2} = \frac{\varkappa^2 \varphi_0'^3}{\varphi_0''^2} = \frac{\varkappa^2}{\varphi_0''^2}, \qquad \varphi_0'' = \varphi''(\eta_0 + 0) \tag{6.4}$$

where ϕ_0'' appears as a new empirical constant; or, using the Reynolds number from the "first approximation" for the laminar region, one may determine ϕ_0'' by requiring continuity in the local Reynolds number at the edge of the laminar region,

$$\frac{l_0^2 u_0'}{\nu} = \frac{\varkappa^2 y_0^2 u_0'}{\nu} = \varkappa^2 \frac{u_0'^3}{\nu u_0''^2} \tag{6.5}$$

so that ($\phi_0' = 1$)

$$\frac{1}{|\varphi_0''|} = \eta_0 \tag{6.6}$$

The second derivative $\phi''(\eta)$ will certainly be subject to a discontinuity at the point $\eta = \eta_0$, but the function $\phi(\eta)$ and its first derivative will preserve their continuity in the whole field of flow. If one agrees to use the approximate equality (6.6), then the further solution will not involve an increase of the number of experimental constants but will keep the same two unknown constants κ and η_0 as in the first approximation.

For the case of uniform motion considered in the present article ($\tau = \tau_w$) we will have, according to the second of the equalities (6.3) when expressed in universal variables,

$$1 = \varphi'\left(\frac{\varkappa^4\varphi'^6}{\varphi''^4} - \frac{\varkappa^4}{\varphi_0''^4} + 1\right)^{1/2} \tag{6.7}$$

From this, solving for ϕ'', we obtain

$$\varphi'' = -\frac{\varkappa\varphi'^2}{[1 + (\varkappa^4/\varphi_0''^4 - 1)\,\varphi'^2]^{1/4}} \tag{6.8}$$

where ϕ_0'' is determined by (6.6), and the minus sign before the fraction on the right-hand side agrees with the condition that the slope of the velocity profile should decrease on moving away from the surface of the body. The constant difference standing in the parentheses under the radical in the denominator is equal, according to (6.6), to

$$\frac{\varkappa^4}{\varphi_0''^4} - 1 = \varkappa^4\eta_0^4 - 1 \tag{6.9}$$

and may be treated as a positive quantity.

Substituting in equation (6.8)

$$\varphi' = \sqrt{\frac{t^4 - 1}{t_0^4 - 1}} \qquad \begin{pmatrix} \varphi' = 1 & \text{for } \eta = \eta_0,\ t = t_0 = \varkappa/\varphi_0'' > 1 \\ \varphi' = 0 & \text{for } \eta = \infty,\ t = 1 \end{pmatrix} \tag{6.10}$$

we obtain the differential equation

$$d\eta = -\frac{2}{\varkappa}\sqrt{t_0^4 - 1}\,\frac{t^4\,dt}{(t^4 - 1)^{3/2}} \tag{6.11}$$

whose integral satisfying the boundary condition

$$t = t_0 \qquad \text{for } \eta = \eta_0$$

is provided by

$$\eta - \eta_0 = +\frac{2}{\varkappa}\sqrt{t_0^4 - 1}\int_t^{t_0}\frac{t^4\,dt}{(t^4 - 1)^{3/2}} \tag{6.12}$$

Evaluation of the integral on the right gives

$$\eta - \eta_0 = \frac{1}{\varkappa}\sqrt{t_0^4 - 1}\left(\frac{t}{\sqrt{t^4 - 1}} - \frac{t_0}{\sqrt{t_0^4 - 1}} + \int_t^{t_0}\frac{dt}{\sqrt{t^4 - 1}}\right) \tag{6.13}$$

The remaining integral on the right-hand side cannot be expressed in terms of elementary functions, but may be rewritten in the form

$$\int_t^{t_0} \frac{dt}{\sqrt{t^4-1}} = \int_1^{t_0} \frac{dt}{\sqrt{t^4-1}} - \int_1^{t} \frac{dt}{\sqrt{t^4-1}} =$$
$$= \frac{1}{\sqrt{2}}\left[F\left(\sec^{-1} t_0; \frac{1}{\sqrt{2}}\right) - F\left(\sec^{-1} t; \frac{1}{\sqrt{2}}\right)\right] \tag{6.14}$$

where the usual notation is used for the elliptic integral of the first kind,

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \tag{6.15}$$

Thus we have

$$\eta = \eta_0 + \frac{1}{\varkappa}\sqrt{t_0^4-1}\left[\frac{t}{\sqrt{t^4-1}} - \frac{t_0}{\sqrt{t^4-1}} + \right.$$
$$\left. + \frac{1}{\sqrt{2}} F\left(\operatorname{arc\,sec} t_0; \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}} F\left(\operatorname{arc\,sec} t; \frac{1}{\sqrt{2}}\right)\right] \tag{6.16}$$

On the other hand, from (6.10) and (6.11) we have

$$d\varphi = \sqrt{\frac{t^4-1}{t_0^4-1}}\, d\eta = -\frac{2}{\varkappa}\frac{t^4 dt}{t^4-1}$$

so that, integrating and taking into account the boundary condition

$$\varphi = \eta_0, \qquad t = t_0 \qquad \text{for } \eta = \eta_0$$

we obtain

$$\varphi = \eta_0 + \frac{1}{2\varkappa}(t - t_0) + \frac{1}{\varkappa}\,[\theta(t_0) - \theta(t)] \tag{6.17}$$

where the notation of (4.14) is retained.

The combined equations (6.16) and (6.17) give the desired velocity profile in the "second approximation". On working out the expressions on the right-hand sides of these equalities near $t = 1$ and eliminating t, we are led to an asymptotic logarithmic formula for the velocity when η is large. In the present article, however, we will content ourselves with the general considerations already presented.

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Translated by D.C.